Choquet Integrals and Multicriteria Decision Making

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February 4, 2008

1 Introduction

A common problem that recurs many times throughout the course of a typical day is making a decision. An entire area of mathematics is devoted to the study of decision making: decision theory. Decision theory aims to model human decision making using mathematics. Additionally, it aims to produce "ideal" decisions, perhaps those that are beyond the computational abilities of humans. Naturally, this is not an easy area of mathematics to develop—decision making is a complex procedure that is difficult to model mathematically. However, several approaches have been taken at solving the problem of mathematically determining the best decision to make in a situation in which several criteria are factors.

In particular, we often need to make decisions that have outcomes that can be affected by several criteria. These are called, descriptively enough, multicriteria decision-making problems.

2 Weighted Sums

The primary idea of this problem of multicriteria decision making is, given a problem with several criteria, to determine which selection of each criterion results in a combination producing the "best" overall solution to the problem. The natural approach to solving this problem is to apply a weight to each criterion to allow each to be a portion of the overall sum of criteria values. The "best" solution, then, has the highest sum of weighted criteria values.

To accomplish this, we define a utility function $u(x)$ to provide a value for the importance of each weighted sum. The user simply provides weights $\alpha_i \in [0, 1]$ that reflect the importance of each criterion $x \in X$. (X is the set of all criteria.) The weights are such that $\sum_{i=1}^{n} \alpha_i = 1$. Then, we define this utility function as follows:

$$u(x) = \sum_{i=1}^{n} \alpha_i \cdot u_i(x_i).$$

2.1 Shortcomings of Weighted Sums

There may exist fundamental differences between the qualities of criteria preventing the addition of their values from producing a new sum value that can be compared to the values comprising the sum. This is essentially a problem of dependencies: two independent criteria can be compared to each other without interfering with each other, but when the two are combined into a new criterion, the resulting value simply "doesn't make sense"—that is, it cannot be compared to other criterion values. One might say it's like comparing apples and oranges.

The main problem here is that when two or more criteria are dependent on each other, it becomes impossible to assign realistic and accurate weighted values of importance to each criterion. For some values of some criteria, other criteria should be weighted differently. Consider, for example, a car. If a car has very poor safety ratings, the maximum speed of the car may be considered completely irrelevant.

3 The Choquet Integral

The Choquet integral, which is a weighted mean that applies meanings to every combination of criteria, can be used instead of the weighted criteria described above. This allows criteria that could generally not be combined independently to be considered together while still maintaining the correct meaning implied by the combination of the criteria.

To discuss the Choquet integral, we must first introduce the notion of a measure. A measure is simply a numeric representation of how desirable the outcome of a decision-making problem will be with some set of criteria. Naturally, then, there will exist some number of measures corresponding to any multicriteria decision-making problem—specifically, there is one measure for each member of the power set representing the problem's criteria.

Having to maintain such a large set of measures, despite high accuracy, leads to inherent complexity problems. The complexity of maintaining these measures is $\mathcal{O}(2^n)$.[1] However, let us ignore this problem of complexity for the time being; it will be addressed soon.

3.1 Non-additive Measures

Mathematically, we can define a non-additive measure as follows: we first denote the set of attributes as I; further, $\mathcal{P}(I)$ is the power set of I. Then, a non-additive measure is a set function $\mu : \mathcal{P}(I) \to [0, 1]$ that satisfies the following three conditions:[2]

- $\mu(\varnothing) = 0$: the empty set has no importance;
- $\mu(I) = 1$: the set of all attributes has maximum importance;
- $\mu(B) \leq \mu(C)$ if $B, C \subset I$ and $B \subset C$: adding a new criterion cannot decrease the importance of a set of criteria.

The fact that the measure is called "non-additive" simply refers to the fact that there exists a one-to-one correspondence between each member of $\mathcal{P}(I)$ and a measure.

3.2 Shapley Values and Interaction Indices

To exploit non-additive measures in a useful way, we must introduce two additional concepts: Shapley values and interaction indices. Shapley values define the importance of criteria independent of other criteria. Interaction indices describe the interaction of criteria with each other within a given set of criteria. Both Shapley values and interaction indices are simply scalar values; however, they can be combined usefully with Choquet integrals to compare criteria while providing the ability to extract semantic meanings applied to groupings of these criteria.

We define Shapley values mathematically as follows, where $\gamma_I(B)$ is defined as $\frac{(|I|-|B|-1)!\cdot|B|!}{|I|!}$ and $|B|$ is defined as the cardinality of the set B:

$$v(j) = \sum_{B \subset I_{\{j\}}} \gamma_I(B) \cdot (\mu(B \cup \{j\}) - \mu(B)).$$

This summation is not difficult to understand:

- The summation takes into account all subsets B of I;
- $\gamma_I(B)$ produces the number of permutations of criteria within the subset B;

[1] This can be inferred, of course, from the fact that the cardinality of a power set $\mathcal{P}(S)$ is 2^n, where n is the cardinality of the set S.

[2] This interval notation $[a, b]$ represents all real numbers between a and b, inclusive.

- the first μ produces the measure of the union of this subset B and criteria j;
- the second μ produces the measure of this subset by itself;
- we subtract the second μ from the first μ to obtain the measure of criteria j by itself, or its "importance."

Interaction indices belong to the interval $[-1, 1]$. Let $I(i, j)$ denote the interaction index between criteria i and j. Then:

- If $I(i, j) > 0$, then i and j are complementary;
- if $I(i, j) < 0$, then i and j are redundant;
- if $I(i, j) = 0$, then i and j are completely independent.

We formally describe interaction indices using the following formula, where $\xi_I(B)$ is defined as $\frac{(|I|-|B|-2)!\cdot|B|!}{(|I|-1)!}$:

$$I(i, j) = \sum_{B \subset I - \{i,j\}} \xi_I(B) \cdot (\mu(B \cup \{i, j\}) - \mu(B \cup \{i\}) - \mu(B \cup \{j\}) + \mu(B)).$$

Again, this summation is not difficult to understand:

- The summation takes into account all subsets B of I, removing criteria i and j;
- $\xi_I(B)$ produces the number of permutations of criteria within the subset B;
- the first μ produces the measure of the union of this subset B and criteria i and j;
- the second μ produces the measure of the union of this subset B and criteria i;
- the third μ produces the measure of the union of this subset B and criteria j;
- the final μ produces the measure of this subset B by itself;
- we subtract the second and third μ from the first μ to obtain the measure of criteria i and j only, or a value indicating how these criteria interact with each other.

3.3 Complexity of Measures

At this point, consider the complexity of calculating the Shapley values and interaction indices of criteria for even a small multicriteria decision-making problem; remember, this problem is $\mathcal{O}(2^n)$. Additionally, it is extremely difficult, if not impossible, to ensure that accurate values are obtained for Shapley values and interaction indices for measures of larger multicriteria decision-making problems.

One way to reduce the complexity of these problems, at the expense of accuracy, of course, is by reducing the cardinality of the subsets for which we maintain measures. Consider the non-additive measure for a moment. We know that we must maintain measures for the power set of the set of criteria—that is, we must consider every possible subset of the set of criteria. Therefore, the largest such subset would contain n items, where n is the number of criteria we are considering.

Instead, we are able to consider only subsets of some cardinality $m < n$. In particular, we use 2-additive measures to reduce complexity. These consider only subsets with cardinality 2 or lower—that is, measures corresponding to sets of only two or fewer criteria.

3.4 2-additive Measures

More formally, a 2-additive measure is a non-additive measure such that all interaction indices of order 3 or higher are null. This is much like considering Shapley values and interaction indices for non-additive measures, but complexity is greatly reduced. As a result, we obtain a slightly simplified mathematical expression used to produce a 2-additive measure. This expression (which is, in fact, a Choquet integral) is as follows:

$$(C)\int_I f\,d\mu = \sum_{I_{ij}>0}(f(i)\wedge f(j))\cdot I_{ij} + \sum_{I_{ij}<0}(f(i)\vee f(j))\cdot |I_{ij}| + \sum_{i=1}^{n}(f(i)\cdot(I_i - \frac{1}{2}\cdot\sum_{j\neq i}|I_{ij}|)).$$

Let us analyze this expression in parts:

- The first summation corresponds to criteria with a positive interaction index (that is, complementary criteria);
- the second summation corresponds to criteria with a negative interaction index (that is, redundant criteria);
- the third (and fourth) summations corresponds to all criteria. Specifically, these summations handle the case of independent criteria.

To reiterate, the reason we use 2-additive measures specifically is because higher-order additive measures are simply too complex to be computed in a reasonable amount of time. 2-additive measures are a compromise between accuracy and complexity.

3.4.1 Application of 2-additive Measures

Let us briefly review the steps required to obtain a 2-additive measure to solve a multicriteria decision-making problem:

1. Define the problem clearly and unambiguously.
2. Determine which criteria are involved with or related to the problem.
3. Determine which subsets of criteria will be considered. Essentially, generate the power set of criteria, then remove all subsets of cardinality 3 or higher.
4. Determine, for each of these subsets:
   - How important this subset is (its Shapley value);
   - how related these criteria are to each other (the subset's interaction index).
5. For each subset, use the Choquet integral provided to obtain a scalar representing how well this subset solves the problem.
6. Note the subset that produces the highest result with the Choquet integral—this one is the optimal solution to the multicriteria decision-making problem.

3.5 Extending Choquet Integrals with Intervals

One problem that becomes immediately obvious when calculating Choquet integrals in the fashion described previously is that there are many, many numbers to keep track of, all of which are presumed to be accurate. In practice, this is rarely the case.

To reduce the impact of this potential lack of precision, it is possible to apply interval arithmetic to Choquet integrals. This essentially leaves Choquet integrals unchanged with only the addition of intervals. The formal expression to describe an interval-based Choquet integral follows:

$$(C_{\mathbb{I}}) \int_I f \, d\mu = \sum_{J_{ij}>0} ((f(i) \vee f(j)) - \frac{1}{2} \cdot (f(i) + f(j))) \cdot J_{ij} + \sum_{J_{ij}<0} ((f(i) \wedge f(j)) - \frac{1}{2} \cdot (f(i) + f(j))) \cdot |J_{ij}| + \sum_{i=1}^{n} f(i) \cdot J_i.$$

Very simply, the user must now only provide J_{ij} and J_i, the interval arithmetic equivalents of I_{ij} and I_i, respectively.

Since this integral uses intervals in its calculation, the result, of course, is another interval.

3.5.1 Shortcomings of Intervals

When the Choquet integral is extended with interval arithmetic, a handful of problems can occur. Most obviously, perhaps, is that there is a perceived reduction of accuracy. This, however, is due to the fact that the input to the integral is now in the form of intervals. Since the input is not exact, one cannot expect the output to be exact, either. However, it is possible that the resulting interval may bound a much larger region than the user expects. This problem, it seems, is inevitable. This is simply an inherent shortcoming of applying intervals to the Choquet integral.

Another problem, however, is not so obvious. When we have several results from Choquet integrals, how do we compare these to each other? Consider, for example, that we have two intervals: $[-5, 5]$ and $[-2, 2]$. Which interval is "better?" Unfortunately, there is no single best answer to this question. Two common approaches are to be an optimist and to be a pessimist: that is, consider only the upper- and lower-bound values of each interval, respectively. However, the best method to apply is highly dependent upon the application.

4 Conclusion

The use of Shapley values along with interaction indices provide a highly accurate but somewhat complex method of solving multicriteria decision-making problems through the use of non-additive measures. The more general process of determining expected utility (which these measures implement) is accomplished using the Choquet integral. Considering only 2-additive measures substantially decreases the complexity of the problem at the potential expense of accuracy. This leads to the question of how accurate the data about the criteria is in the first place. Generally, this data's accuracy is questionable. Therefore, the Choquet integral can be extended to use intervals instead of exact values for Shapley values and interaction indices. However, this introduces the problem of comparing intervals to each other and may further reduce accuracy.

The problem of making a decision in which multiple criteria involved is highly complex. The approach presented herein is a general solution to this problem. However, each individual problem has its own circumstances. It will often be the case that an approach described here will not apply to the problem at hand. Regardless, the Choquet integral, with or without intervals, is a highly useful tool for solving multicriteria decision-making problems.

References

[1] Ceberio, M. and Modave, F., *Interval-Based Multicriteria Decision Making*. 2005.